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On Uniform Global Error Bounds for Convex Inequalities

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Abstract. This paper studies the existence of a uniform global error bound when a convex inequality $g \leqslant 0$, where g is a closed proper convex function, is perturbed. The perturbation neighborhoods are defined by small arbitrary perturbations of the epigraph of its conjugate function. Under certain conditions, it is shown that for sufficiently small arbitrary perturbations the perturbed system is solvable and there exists a uniform global error bound if and only if g satisfies the Slater condition and the solution set is bounded or its recession function satisfies the Slater condition. The results are used to derive lower bounds on the distance to ill-posedness.

Key words: Convex inequality; Global error bound

1. Introduction

Given a feasible system of convex inequalities having a finite global error bound τ and the associated residual function $r(x)$, the Euclidean distance from any point x to the solution set is bounded by $\tau r(x)$. The smallest global error bound can be considered as the condition number of the system. The existence of a finite global error bound and its applications in mathematical programming have been studied extensively [5]. Since real-world problems typically have inaccurate data, it is important to study the behavior of the smallest global error bound when the data of convex inequalities undergo small changes.

The concept of a uniform global error bound or well-conditionedness for a finite system of linear inequalites under small arbitrary perturbations was introduced by Luo and Tseng [4]. Some related results include: The existence of a uniform global error bound for linear inequalities and equalities in Banach spaces [3], for convex inequalities defined by real continuous convex functions [1], for infinite systems of linear inequalities [2], and stability property of a system of inequalities [8].

In this paper we study the existence of a uniform global error bound when a convex inequality $g \leqslant 0$, where g is a closed proper convex function, is perturbed. What are the data of a general closed proper convex function? This question can be answered in a dual way. It is known that a closed proper convex function g is the pointwise supremum of all affine functions h satisfying $h \leqslant g$, and the set of data that defines such affine functions is equal to $\mathrm{epi}(g^*)$, the epigraph of the conjugate function of g. We consider $\mathrm{epi}(g^*)$ as the data of g and our perturbation neighborhoods are defined by small arbitrary perturbations of $\mathrm{epi}(g^*)$. Under certain conditions, it is shown that for sufficiently small arbitrary perturbations the perturbed system is solvable and there exists a uniform global error bound if and only if g satisfies the Slater condition and the solution set is bounded or its recession function satisfies the Slater condition. The results are used to derive lower bounds on the distance to ill-posedness.

2. Notation and preliminaries

Given a convex inequality $g(x) \leq 0$, where g is a closed proper convex function on R^n, let $S(g) = \{x : g(x) \leq 0\}$ be the solution set and $r(x, g) = \max\{g(x), 0\}$ be the residual function. Let $\|\cdot\|$ be the Euclidean norm for a vector and $d(x, C) = \min\{\|x - y\| : y \in C\}$ be the Euclidean distance from x to a nonempty closed convex set C. If $S(g) \neq \emptyset$, the smallest global error bound is defined as

$$\tau(g) = \sup_{x \notin S(g)} \frac{d(x, S(g))}{r(x, g)} .$$

Let g^* be the conjugate function of g, and epi$(\cdot)$ the epigraph of a function. It is known that [7, pp. 102–104] $g(x) = \sup\{(a(u)^T x - b(u)) : u \in U\}$, where U is an index set, $a : U \to R^n$, $b : U \to R^1$, and $\{(a(u), b(u)) : u \in U\} = \mathrm{epi}(g^*)$. Consequently, the convex inequality $g(x) \leq 0$ is closely related to the following infinite system of linear inequalities defined by $\{a, b\}$

$$a(u)^T x - b(u) \leq 0 \text{ for all } u \in U, \text{ where } \{(a(u), b(u)) : u \in U\} = \mathrm{epi}(g^*) .$$

Let $S(a, b) = \{x : a(u)^T x - b(u) \leq 0 \text{ for all } u \in U\}$ denote the solution set, $r(x, \{a, b\}) = \sup\{\max\{a(u)^T x - b(u), 0\} : u \in U\}$ the residual function, and

$$\tau(a, b) = \sup_{x \notin S(a,b)} \frac{d(x, S(a, b))}{r(x, \{a, b\})} .$$

From the formulas given above it follows that $S(a, b) = S(g)$, $r(x, \{a, b\}) = r(x, g)$, and $\tau(a, b) = \tau(g)$.

Let $\|a\|_f = \sup\{\|a(u)\| : u \in U\}$ and $\|b\|_f = \sup\{|b(u)| : u \in U\}$. For $0 \leq \epsilon_1 < \infty$ and $0 \leq \epsilon_2 < \infty$, a perturbation neighborhood of g is defined as

$$B(g, \epsilon_1, \epsilon_2) = \{\tilde{g} : \tilde{g} = \sup\{\tilde{a}(u)^T x - \tilde{b}(u) : u \in U\}\} ,$$

where $\tilde{a} : U \to R^n$, $\tilde{b} : U \to R^1$, $\|\tilde{a} - a\|_f \leq \epsilon_1$, and $\|\tilde{b} - b\|_f \leq \epsilon_2$. A perturbed system $\tilde{g}(x) \leq 0$ is closely related to the infinite system of linear inequalities defined by $\{\tilde{a}, \tilde{b}\}$, $\tilde{a}(u)^T x - \tilde{b}(u) \leq 0$ for all $u \in U$. Thus we can use the following result for infinite systems of linear inequalities [2] to prove the well-conditionedness of a convex inequality.

LEMMA 1. *(a) If there exist $K < \infty$, $\delta > 0$, and x^0 such that $\|x\| \leq K$ for all*

$x \in S(a,b)$ and $a(u)^T x^0 - b(u) \leqslant -\delta$ for all $u \in U$, then for any $0 \leqslant \beta < 1$, $\|\tilde{a} - a\|_f \leqslant \beta\delta/(2K)$, $\|\tilde{b} - b\|_f \leqslant \beta\delta/2$, it holds that $S(\tilde{a}, \tilde{b}) \neq \emptyset$ and

$$\tau(\tilde{a}, \tilde{b}) \leqslant 2(1-\beta)^{-1}\delta^{-1}\frac{K + \tau(a,b)\epsilon_2}{1 - \tau(a,b)\epsilon_1} \leqslant 2(1-\beta)^{-1}\delta^{-1}\frac{K + 2\delta^{-1}K\epsilon_2}{1 - 2\delta^{-1}K\epsilon_1}.$$

Furthermore, for all $\tilde{x} \in S(\tilde{a}, \tilde{b})$, it holds that

$$\|\tilde{x}\| \leqslant \frac{K + \tau(a,b)\epsilon_2}{1 - \tau(a,b)\epsilon_1} \leqslant \frac{K + 2\delta^{-1}K\epsilon_2}{1 - 2\delta^{-1}K\epsilon_1}.$$

(b) If $S(a,b)$ is unbounded and there exist a unit vector w and $\eta > 0$ such that $a(u)^T w \leqslant -\eta$ for all $u \in U$, then $\tau(a,b) \leqslant \eta^{-1}$. □

Note that $a(U) = \{a(u) : u \in U\}$ and $b(U) = \{b(u) : u \in U\}$ are convex sets, but not necessarily compact. The results of [2] that use the compactness of $a(U)$ and $b(U)$ cannot be applied here.

For a proper convex function f, let f^∞ denote the recession function and $\mathrm{dom}(f) = \{x : f(x) < \infty\}$ the domain of finiteness. It is easy to see that $a(U) = \mathrm{dom}(g^*)$. Using [7, Theorem 9.4], one can verify that for all $0 < \epsilon_1 < \infty$, $0 < \epsilon_2 < \infty$, and $\tilde{g} \in B(g, \epsilon_1, \epsilon_2)$, it holds that $\tilde{g}$ is a closed proper convex function and $\mathrm{dom}(\tilde{g}) = \mathrm{dom}(g)$.

As in [4], we say that the system $g \leqslant 0$ is well-conditioned under perturbations $B(g, \epsilon_1, \epsilon_2)$ if for all $\tilde{g} \in B(g, \epsilon_1, \epsilon_2)$, the perturbed system $\tilde{g} \leqslant 0$ is solvable, $\tau(\tilde{g})$ is finite and uniformly bounded.

3. The main results

LEMMA 2. *If there exist $K < \infty$, $\delta > 0$, and x^0 such that $\|x\| \leqslant K$ for all $x \in S(g)$ and $g(x^0) \leqslant -\delta < 0$, then for any $0 \leqslant \beta < 1$, $0 \leqslant \epsilon_1 \leqslant \beta\delta/(2K)$, $0 \leqslant \epsilon_2 \leqslant \beta\delta/2$, and $\tilde{g} \in B(g, \epsilon_1, \epsilon_2)$, it holds that $S(\tilde{g}) \neq \emptyset$ and*

$$\tau(\tilde{g}) \leqslant 2(1-\beta)^{-1}\delta^{-1}\frac{K + \tau(g)\epsilon_2}{1 - \tau(g)\epsilon_1} \leqslant 2(1-\beta)^{-1}\delta^{-1}\frac{K + 2\delta^{-1}K\epsilon_2}{1 - 2\delta^{-1}K\epsilon_1}.$$

Furthermore, for all $\tilde{x} \in S(\tilde{g})$, it holds that

$$\|\tilde{x}\| \leqslant \frac{K + \tau(g)\epsilon_2}{1 - \tau(g)\epsilon_1} \leqslant \frac{K + 2\delta^{-1}K\epsilon_2}{1 - 2\delta^{-1}K\epsilon_1}.$$

Proof. As $g(x^0) = \sup\{(a(u)^T x^0 - b(u)) : u \in U\}$, we have that $a(u)^T x^0 - b(u) \leqslant g(x^0) \leqslant -\delta < 0$ for all $u \in U$. Since $S(\tilde{g}) = S(\tilde{a}, \tilde{b})$, $r(x, \tilde{g}) = r(x, \{\tilde{a}, \tilde{b}\})$, and $\tau(\tilde{g}) = \tau(\tilde{a}, \tilde{b})$ for all $\tilde{g} \in B(g, \epsilon_1, \epsilon_2)$, the result follows from Lemma 1 (a). □

LEMMA 3. *If there exist a unit vector w and $\eta > 0$ such that $g^\infty(w) \leqslant -\eta < 0$, then* *for all* $0 \leqslant \epsilon_1 < \eta$, $0 \leqslant \epsilon_2 < \infty$, *and* $\tilde{g} \in B(g, \epsilon_1, \epsilon_2)$, *it holds that* $S(\tilde{g})$ *is unbounded and* $\tau(\tilde{g}) \leqslant (\eta - \epsilon_1)^{-1}$.

Proof. From [7, Theorems 8.5 and 9.4], the recession function of $g(x) = \sup\{a(u)^T x - b(u) : u \in U\}$ is $g^\infty(x) = \sup\{a(u)^T x : u \in U\}$. Hence, $a(u)^T w \leqslant g^\infty(w) \leqslant -\eta < 0$ for all $u \in U$. Let $\tilde{g}(x) = \sup\{\tilde{a}(u)^T x - \tilde{b}(u) : u \in U\} \in B(g, \epsilon_1, \epsilon_2)$. For all $u \in U$, $|\tilde{a}(u)^T w - a(u)^T w| \leqslant \|\tilde{a} - a\|_f \leqslant \epsilon_1$ and thus $\tilde{a}(u)^T w \leqslant a(u)^T w + \epsilon_1 \leqslant -\eta + \epsilon_1 < 0$. Therefore, $\tilde{g}^\infty(w) \leqslant -\eta + \epsilon_1 < 0$. As $\tilde{g}$ is a closed proper convex function, $\tilde{g}^\infty(w) < 0$ implies that $S(\tilde{g})$ is unbounded. Applying Lemma 1 (b) to $\{\tilde{a}, \tilde{b}\}$, we have that $\tau(\tilde{a}, \tilde{b}) \leqslant (\eta - \epsilon_1)^{-1}$. Lemma 3 then follows from the fact $\tau(\tilde{g}) = \tau(\tilde{a}, \tilde{b})$. □

What has not been discussed is the case that g satisfies the Slater condition, g^∞ does not satisfy the Slater condition, and $S(g)$ is unbounded. Under the condition that dom(g^*) is closed, we can construct, as in [4, Theorem 2.4], a set of perturbations g^ϵ and show that $\tau(g^\epsilon)$ cannot be uniformly bounded as ϵ approaches zero.

THEOREM 1. *If dom(g^*) is closed, then the following statements are equivalent.*

(a) Either g satisfies the Slater condition and $S(g)$ is bounded or g^∞ satisfies the Slater condition.

(b) There exist $0 < \epsilon_1 < \infty$ and $0 < \epsilon_2 < \infty$ such that g is well-conditioned under perturbations $B(g, \epsilon_1, \epsilon_2)$.

Proof. (a) $\Rightarrow$ (b). As g^∞ is positive homogeneous, the results follows directly form Lemmas 2 and 3.

(b) $\Rightarrow$ (a). Choosing $\tilde{g}(x) = \sup\{a(u)^T x - (b(u) - \epsilon_2) : u \in u\} \in B(g, \epsilon_1, \epsilon_2)$ and $x^0 \in S(\tilde{g})$, we have $g(x^0) \leqslant -\epsilon_2 < 0$. If $S(g)$ is bounded, then (a) is valid. Now suppose that, on the contrary, $S(g)$ is unbounded and $\{x : g^\infty(x) < 0\} = \emptyset$. It follows from [7, Theorem 27.1] and the closedness of dom(g^*) that $0 \in a(U)$, i.e., $a(u^0) = 0$ for some $u^0 \in U$. In the rest of the proof, we construct a set of perturbations g^ϵ and show that $\tau(g^\epsilon)$ cannot be uniformly bounded as ϵ approaches zero. Since $S(g)$ is closed and unbounded, there exists a unit vector z satisfying $a(u)^T z \leqslant 0$ for all $u \in U$. For any ϵ satisfying $1/\epsilon > |z^T x^0|$, define

$$a^\epsilon(u) = \begin{cases} a(u) + \epsilon b(u) z, & \text{if } u = u^0 ; \\ a(u), & \text{if } u \in U \backslash \{u^0\}, \end{cases}$$

$g^\epsilon(x) = \sup\{a^\epsilon(u)^T x - b(u) : u \in U\}$, $\alpha = 1/\epsilon - z^T x^0$, and $x^\epsilon = x^0 + \alpha z$. One can verify that

$$a^\epsilon(u)^T x^\epsilon - b(u) = \begin{cases} 0, & \text{if } u = u^0 ; \\ a(u)^T x^0 - b(u) + \alpha a(u)^T z < 0, & \text{if } u \in U \backslash \{u^0\}. \end{cases} \quad (1)$$

Therefore, x^ϵ is a boundary point of $S(a^\epsilon, b)$ and $a^\epsilon(u^0)^T x - b(u^0) \leqslant 0$ is the binding constraint. As $g(x^0) \leqslant -\epsilon_2$ and $a(u^0)=0$, we have $a(u^0)^T x^0 - b(u^0) = -b(u^0) \leqslant g(x^0) \leqslant -\epsilon_2$ and thus $b(u^0) \geqslant \epsilon_2 > 0$. Hence, $a^\epsilon(u^0) = \epsilon b(u^0) z$ and z is the unit normal vector to the binding constraint at the boundary point x^ϵ. Consequently, the projection of $x^\epsilon + \alpha z$ onto $S(a^\epsilon, b)$ is x^ϵ and $d(x^\epsilon + \alpha z, S(a^\epsilon, b)) = \alpha$. On the other hand, one can verify, using (1), that

$$a^\epsilon(u)^T(x^\epsilon + \alpha z) - b(u) = \begin{cases} (1-\epsilon z^T x^0) b(u^0)\,, & \text{if } u = u^0\,; \\ a(u)^T x^0 - b(u) + 2\alpha a(u)^T z \leqslant 0\,, & \text{if } u \in U \backslash \{u^0\}\,. \end{cases}$$

Hence, $r(x^\epsilon + \alpha z, \{a^\epsilon, b\}) \leqslant 2b(u^0)$. As $\epsilon \to 0$, we have $\|a^\epsilon - a\|_f \to 0$, $d(x^\epsilon + \alpha z, S(a^\epsilon, b)) \to \infty$, and $r(x^\epsilon + \alpha z, \{a^\epsilon, b\}) \leqslant 2b(u^0)$. Therefore $\tau(a^\epsilon, b) = \tau(g^\epsilon)$ cannot be uniformly bounded as ϵ approaches zero. The contradiction shows that $\{x : g^\infty(x) < 0\} \neq \emptyset$. □

In particular, if g is a polyhedral convex function, then $\mathrm{dom}(g^*)$ is closed and Theorem 1 reduces to [4, Theorem 3.3].

Finally, we use Lemmas 2 and 3 to derive lower bounds on the distance to ill-posedness for well-conditioned systems.

Let $P = \{\bar g : S(\bar g) = \emptyset\}$, where $\bar g(x) = \sup\{\bar a(u)^T x - \bar b(u) : u \in U\}$, $\bar a : U \to R^n$, $\bar b : U \to R^1$, $\|\bar a - a\|_f < \infty$, and $\|\bar b - b\|_f < \infty$. The distance to ill-posedness for $g \not\in P$ is defined as [6]

$$\mathrm{dist}(g, P) = \inf\{\max\{\|a - \bar a\|_f, \|b - \bar b\|_f\} : \bar g \in P\}\,.$$

THEOREM 2. *(a) If $\|x\| \leqslant K$ for all $x \in S(g)$ for some finite K and there exist x^0 and $\delta > 0$ such that $g(x^0) \leqslant -\delta < 0$, then dist$(g, P) \geqslant \min\{\delta/(2K), \delta/2\}$.*

(b) If $S(g)$ is unbounded and there exist a unit vector w^0 and $\eta > 0$ such that $g^\infty(w^0) \leqslant -\eta < 0$, then dist$(g, P) \geqslant \eta$.

Proof. (a) Let $0 < \beta < 1$ and $\bar g \in P$. It follows from Lemma 2 that either $\|a - \bar a\|_f > \beta\delta/(2K)$ or $\|b - \bar b\|_f > \beta\delta/2$, which implies that

$$\max\{\|a - \bar a\|_f, \|b - \bar b\|_f\} > \min\{\beta\delta/(2K), \beta\delta/2\}\,. \tag{2}$$

Since (2) holds for any $\bar g \in P$, we have

$$\mathrm{dist}(g, P) \geqslant \min\{\beta\delta/(2K), \beta\delta/2\}\,. \tag{3}$$

Since (3) holds for any $0 < \beta < 1$, we have $\mathrm{dist}(g, P) \geqslant \min\{\delta/(2K), \delta/2\}$. The proof of (b) is similar to that of (a) and is omitted. □

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